Weyl's spaces with shear-free and expansion-free conformal Killing vectors and the motion of a free spinless test particle.

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Abstract

Conditions for the existence of shear-free and expansion-free non-null vector fields in spaces with affine connections and metrics are found. On their basis Weyl's spaces with shear-free and expansion-free conformal Killing vectors are considered. The necessary and sufficient conditions are found under which a free spinless test particle could move in spaces with affine connections and metrics on a curve described by means of an auto-parallel equation. In Weyl's spaces with Weyl's covector, constructed by the use of a dilaton field, the dilaton field appears as a scaling factor for the rest mass density of the test particle.

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1 Introduction

In the last years Weyl's spaces have deserved some interest related to the possibility of using mathematical models of space-time different from (pseudo) Riemannian spaces without torsion (V_n-spaces) or with torsion (U_n-spaces)[1] $\div$ [4]. On the one side, Weyl's spaces appear as a generalization of V_n- and U_n-spaces. On the other side, they are special cases of spaces with affine connections and metrics. The use of spaces with affine connections and metrics as models of space-time has been critically evaluated from different points of view [5], [6]. But recently, it has been proved that in spaces with contravariant and covariant

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affine connections (whose components differ only by sign) and metrics [(L_n, g)-spaces] as well as in spaces with contravariant and covariant affine connections (whose components differ not only by sign) and metrics [$(\overline{L}_n, g)$-spaces] [7], [8] the principle of equivalence holds [9]$\div$[16]. In these spaces special types of transports (called Fermi-Walker transports) [17] $\div$ [19] exist which do not deform a Lorentz basis. Therefore, the low of causality is not abused in (L_n, g)- and $(\overline{L}_n, g)$-spaces if one uses a Fermi-Walker transport instead of a parallel transport (used in a V_n-space). Moreover, there also exist other types of transports (called conformal transports) [20], [21] under which a light cone does not deform. At the same time, the auto-parallel equation can play the same role in (L_n, g)- and $(\overline{L}_n, g)$-spaces as the geodesic equation does in the Einstein theory of gravitation (ETG) in V_n-spaces [22], [23]. On this basis, many of the differential-geometric construction used in the ETG in V_4-spaces could be generalized for the cases of (L_n, g)- and $(\overline{L}_n, g)$-spaces, and especially for Weyl's spaces without torsion (W_n or $\overline{W}_n$-spaces) or in Weyl's spaces with torsion (Y_n- or $\overline{Y}_n$-spaces) as special cases of (L_n, g)- or $(\overline{L}_n, g)$-spaces. Bearing in mind this background a question arises about possible physical applications and interpretation of mathematical constructions from ETG generalized for Weyl's spaces. It is well known that every classical field theory over spaces with affine connections and metrics could be considered as a theory of continuous media in these spaces [24] $\div$ [27]. On this ground, notions of the continuous media mechanics (such as deformation velocity and acceleration, shear velocity and acceleration, rotation velocity and acceleration, expansion velocity and acceleration) have been used as invariant characteristics for spaces admitting vector fields with special kinematic characteristics [28], [29].

1.1 Problems and results

The main task of this paper is the investigation of Weyl's spaces with respect to their ability to admit conformal contravariant Killing vector fields that are shear-free and expansion-free vector fields. On this basis, conditions for the existence of shear-free and expansion-free non-null (non-isotropic) vector fields in spaces with affine connections and metrics are found and then specialized for Weyl's spaces. At the same time, a possible interpretation of a dilaton field, appearing in the structure of special types of Weyl's spaces, is found on the basis of the auto-parallel equation describing the motion of a free spinless test particle in these types of spaces.

In Section 2 some recurrent relations in spaces with affine connections and metrics are considered. The equivalence of the action of the Lie differential operator and of the covariant differential operator on the invariant volume element is introduced as condition for the metrics and for a vector field along which these operators act. In Section 3 some properties of Weyl's spaces are recalled related to the properties of scalar and tensor fields over such type of spaces. In Section 4 the notion of the relative velocity and its kinematic characteristics in spaces with affine connections and metrics related to the sear, rotation, and expansion velocities are introduced. It is shown that the equivalence condition,

proposed in Section 2, appears as a condition for the existence of shear-free and expansion-free non-null vector fields in spaces with affine connections and metrics. The same condition in a Weyl's space appears as a condition for the existence of a shear-free and expansion-free conformal Killing vector field of special type. Sufficient conditions are found under which in special type of Weyl's spaces non-null auto-parallel, shear-free, and expansion-free conformal Killing vector fields could exist. In Section 5 the auto-parallel equation in Weyl's spaces is discussed as an equation for describing the motion of a free moving spinless test particle. Concluding remarks comprise the final Section 6. The most considerations are given in details (even in full details) for those readers who are not familiar with the investigated problems.

1.2 Abbreviations, definitions, and symbols

In the further considerations in this paper we will use the following abbreviations, definitions and symbols:

$:=$ means by definition.

M is a symbol for a differentiable manifold with $\dim M = n$. $T(M) := \cup_{x\in M} T_x(M)$ and $T^*(M) := \cup_{x\in M} T^*_x(M)$ are the tangent and the cotangent spaces at M respectively.

$(\overline{L}_n, g)$, $\overline{Y}_n$, $\overline{U}_n$, and $\overline{V}_n$ are spaces with contravariant and covariant affine connections and metrics whose components *differ not only by sign* [7]. In such type of spaces the non-canonical contraction operator S acts on a contravariant basic vector field e_j (or ∂_j)$\in \{e_j$ (or $\partial_j)\} \subset T(M)$ and on a covariant basic vector field e^i (or dx^i) $\in \{e^i$ (or $dx^i)\} \subset T^*(M)$ in the form

$$\begin{array}{rcllll} & S & : & (e^i, e_j) \longrightarrow S(e^i, e_j) := S(e_j, e^i) := f^i{}_j \ , & & \\ & f^i{}_j & \in & C^r(M) \ , & r \geqq 2 \ , & \det(f^i{}_j) \neq 0, \\ \exists & f_i{}^k & \in & C^r(M), & r \geqq 2 \ : & f^i{}_j \cdot f_i{}^k := g^k_j , \end{array}$$

In these spaces, for example, $g(u) = g_{ik} \cdot f^k{}_j \cdot u^j \cdot dx^i := g_{i\overline{j}} \cdot u^j \cdot dx^i = g_{ij} \cdot u^{\overline{j}} \cdot dx^i := u_i \cdot dx^i$, $g(u,u) = g_{kl} \cdot f^k{}_i \cdot f^l{}_j \cdot u^i \cdot u^j := g_{\overline{i}\overline{j}} \cdot u^i \cdot u^j = g_{ij} \cdot u^{\overline{i}} \cdot u^{\overline{j}} = u_j \cdot u^{\overline{j}} := u_{\overline{i}} \cdot u^i$, $g^{\overline{i}\overline{j}} \cdot g_{jk} = \delta^i_k = g^i_k$, $g_{\overline{i}\overline{k}} . g^{kj} = g^j_i$. The components $\delta^i_j := g^i_j$ ($|= 0$ for $i \neq j$ and $|= 1$ for $i = j$) are the components of the Kronecker tensor $Kr := g^i_j \cdot \partial_i \otimes dx^j$.

(L_n, g), Y_n, U_n, and V_n are spaces with contravariant and covariant affine connections and metrics whose components *differ only by sign* [30], [31]. In such type of spaces the canonical contraction operator $S := C$ acts on a contravariant basic vector field e_j (or ∂_j)$\in \{e_j$ (or $\partial_j)\} \subset T(M)$ and on a covariant basic vector field e^i (or dx^i) $\in \{e^i$ (or $dx^i)\} \subset T^*(M)$ in the form

$$C : (e^i, e_j) \longrightarrow C(e^i, e_j) := C(e_j, e^i) := \delta^i_j := g^i_j \ .$$

In these spaces, for example, $g(u) = g_{ik} \cdot g^k_j \cdot u^j \cdot dx^i := g_{ij} \cdot u^j \cdot dx^i = u_i \cdot dx^i$, $g(u,u) = g_{kl} \cdot g^k_i \cdot g^l_j \cdot u^i \cdot u^j := g_{ij} \cdot u^i \cdot u^j = u_i \cdot u^i$.

Remark. All results found for $(\overline{L}_n, g)$-spaces could be specialized for (L_n, g)-spaces by omitting the bars above or under the indices.

∇_u is the covariant differential operator acting on the elements of the tensor algebra $\mathcal{T}$ over M. The action of ∇_u is called covariant differentiation (covariant transport) along a contravariant vector field u, for instance,

$$\nabla_u v := v^i{}_{;j} \cdot u^j \cdot \partial_i = (v^i{}_{,j} + \Gamma^i_{kj} \cdot v^k) \cdot u^j \cdot \partial_i \ , \qquad v \in T(M) \ , \tag{1}$$

where $v^i{}_{,j} := \partial v^i / \partial x^j$ and Γ^i_{jk} are the components of the contravariant affine connection Γ in a contravariant co-ordinate basis $\{\partial_i\}$. The result $\nabla_u v$ of the action of ∇_u on a tensor field $v \in \otimes^k_l(M)$ is called covariant derivative of v along u. For covariant vectors and tensor fields an analogous relation holds, for instance,

$$\nabla_u w = w_{i;j} \cdot u^j \cdot dx^i = (w_{i,j} + P^l_{ij}.w_l) \cdot u^j \cdot .dx^i \quad , \qquad w \in T^*(M) \ . \tag{2}$$

where P^l_{ij} are the components of the covariant affine connection P in a covariant co-ordinate basis $\{dx^i\}$. For (L_n, g), Y_n, W_n, U_n, and V_n-spaces $P^l_{ij} = -\Gamma^l_{ij}$.

$\pounds_u$ is the Lie differential operator [7] acting on the elements of the tensor algebra $\mathcal{T}$ over M. The action of $\pounds_u$ is called dragging-along a contravariant vector field u. The result $\pounds_u v$ of the action of $\pounds_u$ on a tensor field v is called Lie derivative of v along u.

The n-form $d\omega := \frac{1}{n!} \cdot \sqrt{-d_g} \cdot \varepsilon_{i_1 \ldots i_n} \cdot dx^{i_1} \wedge \ldots \wedge x^{i_n}$, where $d_g := \det(g_{ij}) < 0$, $\varepsilon_{i_1 \ldots i_n}$ are the components of the full antisymmetric Levi-Civita symbol, is called invariant volume element in M.

The result of the action of the covariant differential operator ∇_u on the invariant volume element $d\omega$ can be written in the form [7],

$$\nabla_u(d\omega) = \frac{1}{2} \cdot \overline{g}[\nabla_u g] \cdot d\omega = \frac{1}{2} \cdot g^{\overline{ij}} \cdot g_{ij;k} \cdot u^k \cdot d\omega \quad . \tag{3}$$

The result of the action of the Lie differential operator $\pounds_u$ on the invariant volume element $d\omega$ can be written in the form [7]

$$\pounds_u(d\omega) = \frac{1}{2} \cdot \overline{g}[\pounds_u g] \cdot d\omega = \frac{1}{2} \cdot g^{\overline{ij}} \cdot \pounds_u g_{ij} \cdot d\omega \quad . \tag{4}$$

Let us recall some well known facts about differential-geometric structures over spaces with affine connections and metrics.

2 Recurrent relations in spaces with affine connections and metrics

1. A parallel transport of a contravariant vector field $\xi \in T(M)$ along the vector field u could be defined in the form

$$\nabla_u \xi = f \cdot \xi \ , \qquad f \in C^r(M) \ . \tag{5}$$

An equation of this type is called recurrent equation (or recurrent relation for the vector field ξ) [32], [23]

A special case of a parallel transport is the auto-parallel transport of a contravariant vector field $u \in T(M)$ along itself

$$\nabla_u u = k \cdot u \ , \qquad k \in C^r(M) \ . \tag{6}$$

The recurrent relation (6) is the auto-parallel equation for u in its non-canonical form. After changing the parameter of the curve on which u is a tangent vector the auto-parallel equation could be written in its canonical form as [23]

$$\nabla_{\overline{u}} \overline{u} = 0 \quad . \tag{7}$$

On the other side, after contracting (6) with $g(u)$, it follows that

$$\begin{aligned} g(u, \nabla_u u) &= k \cdot g(u,u) := k \cdot e \quad , \qquad g(u,u) := e :\neq 0 \quad , \\ k &= \frac{1}{e} \cdot g(u, \nabla_u u) = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u,u)] \quad . \end{aligned} \tag{8}$$

For (pseudo) Riemannian spaces $[\nabla_u g = 0$ for $\forall u \in T(M)]$ and normalized vector field u ($e :=$const., $ue = 0$) the function k is equal to zero and $\nabla_u u = 0$.

2. The results of the action of the covariant differential operator ∇_u and of the Lie differential operator $\pounds_u$ on the invariant volume element $d\omega$ are recurrent relations for $d\omega$

$$\nabla_u (d\omega) = \overline{f}_u \cdot d\omega \quad , \qquad \overline{f}_u := \frac{1}{2} \cdot \overline{g}[\nabla_u g] \in \otimes^0_0(M) \subset C^r(M) \ , \tag{9}$$

$$\pounds_u (d\omega) = \widetilde{f}_u \cdot d\omega \quad , \qquad \widetilde{f}_u := \frac{1}{2} \cdot \overline{g}[\pounds_u g] \in \otimes^0_0(M) \subset C^r(M) \ . \tag{10}$$

3. A conformal Killing vector field is defined by analogous type of a recurrent equation.

Definition 1. A conformal Killing vector field is a contravariant vector field u obeying the equation

$$\pounds_u g = \lambda_u \cdot g \quad , \qquad \lambda_u \in \otimes^0_0(M) \subset C^r(M) \ . \tag{11}$$

The equation is called conformal Killing equation. After contracting $\pounds_u g$ and g from the last equation with $\overline{g} = g^{ij} \cdot \partial_i . \partial_j$ $[\partial_i . \partial_j := \frac{1}{2} \cdot (\partial_i \otimes \partial_j + \partial_j \otimes \partial_i)]$ by both basic vector fields, i.e. after finding the relations

$$\overline{g}[\pounds_u g] = g^{\overline{kl}} \cdot \pounds_u g_{ij} \qquad \text{and} \qquad \overline{g}[g] = g^{\overline{kl}} . g_{kl} = n = \dim M \quad , \tag{12}$$

it follows for λ_u

$$\lambda_u = \frac{1}{n} \cdot \overline{g}[\pounds_u g] = \frac{2}{n} \cdot \widetilde{f}_u \ . \tag{13}$$

Therefore, the factor λ_u in a conformal Killing vector equation is related to the factor $\widetilde{f}_u$ by which the invariant volume element changes when dragged along the same vector field.

4. Let us now consider the condition for the existence of a Weyl's space.

Definition 2. A Weyl's space is a differentiable manifold M with $\dim M := n$, provided with affine connections Γ and P (with $P \neq -\Gamma$ or $P = -\Gamma$) and a metric g with covariant derivative of g along an arbitrary given contravariant vector field $u \in T(M)$ in the form

$$\nabla_u g := \frac{1}{n} \cdot Q_u \cdot g \ . \tag{14}$$

The existence condition is a recurrent relation for the metric g. Here

$$\begin{aligned} Q_u &= Q_j \cdot u^j \ , \qquad Q := Q_j \cdot dx^j \ , \\ Q_j &: = \underline{Q}_k \cdot f^k{}_j := \underline{Q}_{\overline{j}} \text{ for } P \neq -\Gamma \\ Q_j &\equiv Q_j \text{ for } P = -\Gamma \ . \end{aligned} \tag{15}$$

The covariant vector field (1-form) $\overline{Q} := \frac{1}{n} \cdot Q_j \cdot dx^j = \frac{1}{n} \cdot Q$ is called Weyl's covariant (covector) field. If Q is an exact form, i. e. if $Q = -d\overline{\varphi} = -\overline{\varphi}_{,j} \cdot dx^j$ with $Q_j = -\overline{\varphi}_{,j}$, $\overline{\varphi} \in C^r(M)$, $r \geqq 2$, then for a contravariant vector field $u := d/d\tau$ the invariant Q_u could be written in the form $Q_u = -d\overline{\varphi}/d\tau$. The scalar field $\overline{\varphi}$ is called dilaton field. The reason for this notation follows from the properties of the Weyl's spaces considered below.

After contracting $\nabla_u g$ and g from the last equation with $\overline{g} = g^{ij} \cdot \partial_i . \partial_j$ by both basic vector fields, i.e. after finding the relations

$$\overline{g}[\nabla_u g] = g^{\overline{ij}} \cdot g_{ij;k} \cdot u^k \qquad \text{and} \qquad \overline{g}[g] = g^{\overline{kl}} . g_{kl} = n = \dim M \ , \tag{16}$$

it follows for Q_u

$$Q_u = \overline{g}[\nabla_u g] = 2 \cdot \ \overline{f}_u \ , \qquad \overline{f}_u = \frac{1}{2} \cdot Q_u \qquad . \tag{17}$$

Therefore, for Weyl's spaces, we have the recurrent relation for the invariant volume element $d\omega$

$$\nabla_u (d\omega) = \frac{1}{2} \cdot Q_u . d\omega \qquad . \tag{18}$$

5. If we now compare the recurrent relations obtained as a result of the action of the covariant differential operator ∇_u and of the Lie differential operator $\pounds_u$ on the invariant volume element $d\omega$ the question could arise what are the conditions for the equivalence of the action of both the differential operators on $d\omega$, i.e. under which conditions for the vector field u and the metric g we could have the relation

$$\nabla_u (d\omega) = \pounds_u (d\omega) \ , \tag{19}$$

which is equivalent to the relation

$$\overline{g}[\nabla_u g] = \overline{g}[\pounds_u g] \qquad \text{or} \qquad \overline{g}[\nabla_u g - \pounds_u g] = 0 \ \ . \tag{20}$$

What does this relation physically mean? The action of the covariant differential operator ∇_u is determined only on a curve at which u is a tangent vector. The transport of g is only on the curve and not on the vicinities out of the points of the curve. The action of the Lie differential operator is determined on the vicinities on and out of the points of the curve with tangent vector u on it. The dragging-along of g is on the whole vicinities of the points of the curve and not only along the curve. If the dragging-along u of g is equal to the transport of g along u, then an observer with its worldline as the curve with tangent vector u could not observe any changes in its worldline vicinity different from those who could register on its worldline. The observer will see its surroundings as if they are moving with him.

It is obvious that in the general case, in $(\overline{L}_n, g)$-spaces, a sufficient condition for fulfilling the last relation is the condition

$$\nabla_u g - \pounds_u g = 0 \ \ . \tag{21}$$

The last condition is fulfill:

(a) in Riemannian spaces (with or without torsion) [for which $\nabla_u g = 0$ for $\forall u \in T(M)$], when the Killing equation [33]

$$\pounds_u g = 0 \tag{22}$$

is fulfilled for the vector field u.

(b) in Weyl's spaces (with or without torsion) [for which $\nabla_u g = \frac{1}{n} \cdot Q_u \cdot g$ for $\forall u \in T(M)$], when the conformal Killing equation

$$\pounds_u g = \lambda_u \cdot g \qquad \text{with} \qquad \lambda_u = \frac{1}{n} \cdot Q_u \tag{23}$$

is fulfilled for the vector field u.

In $(\overline{L}_n, g)$-spaces, the relation $\nabla_u g - \pounds_u g = 0$ could be written in a coordinate basis in the form

$$\begin{aligned} \pounds_u g_{ij} &= g_{ij;k} \cdot u^k + g_{kj} \cdot u^{\overline{k}}{}_{;\underline{i}} + g_{ik} \cdot u^{\overline{k}}{}_{;\underline{j}} + (g_{kj} \cdot T_{l\underline{i}}{}^{\overline{k}} + g_{ik} \cdot T_{l\underline{j}}{}^{\overline{k}}) \cdot u^l = \\ &= g_{ij;k} \cdot u^k \ , \end{aligned} \tag{24}$$

or in the forms

$$g_{kj} \cdot u^{\overline{k}}{}_{;\underline{i}} + g_{ik} \cdot u^{\overline{k}}{}_{;\underline{j}} + (g_{kj} \cdot T_{l\underline{i}}{}^{\overline{k}} + g_{ik} \cdot T_{l\underline{j}}{}^{\overline{k}}) \cdot u^l = 0 \ \ , \tag{25}$$

$$g_{kj} \cdot (u^{\overline{k}}{}_{;\underline{i}} - T_{\underline{i}l}{}^{\overline{k}} \cdot u^l) + g_{ik} \cdot (u^{\overline{k}}{}_{;\underline{j}} - T_{\underline{j}l}{}^{\overline{k}} \cdot u^l) = 0 \ , \tag{26}$$

where

$$T_{\underline{i}l}{}^{\overline{k}} := f_i{}^m \cdot T_{ml}{}^n \cdot f^k{}_n, \qquad\qquad u^{\overline{k}}{}_{;\underline{j}} := f^k{}_l \cdot u^l{}_{;m} \cdot f_j{}^m.$$

The equation (26) could be called "generalized conformal Killing equation" for the vector field u in the case $\pounds_u g = \nabla_u g$.

After multiplication of the last expression, equivalent to $\pounds_u g_{ij} = g_{ij;k} \cdot u^k$, with $u^{\overline{i}}$ and $g^{m\overline{j}}$ and summation over $\overline{i}$ and $\overline{j}$ (and then changing the index m with i) we can find the equation for the vector field u in a co-ordinate basis

$$u^i{}_{;j} \cdot u^j + g_{\overline{lk}} \cdot u^l \cdot (u^k{}_{;j} - T_{jm}{}^k \cdot u^m) \cdot g^{ji} = 0 \quad , \tag{27}$$

or as an equation for the acceleration $a = a^i \cdot \partial_i$ with

$$a^i = -g_{\overline{lk}} \cdot u^l \cdot (u^k{}_{;j} - T_{jm}{}^k \cdot u^m) \cdot g^{ji} = -g_{\overline{lk}} \cdot u^l \cdot k^{ki} \quad , \tag{28}$$

where

$$k^{ki} = (u^k{}_{;j} - T_{jm}{}^k \cdot u^m) \cdot g^{ji} \ . \tag{29}$$

On the other side, from the equation (26) it is obvious that a sufficient condition for fulfilling the equation (26) is the condition for u^i

$$u^k{}_{;j} - T_{jl}{}^k \cdot u^l = 0 \qquad \text{or} \qquad u^k{}_{;j} = T_{jl}{}^k \cdot u^l \quad . \tag{30}$$

From the last expression, it follows that if the vector u fulfills this condition it should be an auto-parallel vector field since

$$u^i{}_{;j} \cdot u^j = a^i = 0 \quad . \tag{31}$$

If (30) is fulfilled, then the following relations are valid:

$$\begin{aligned} u^m \cdot R^i{}_{mkl} &= -(T_{ml}{}^i{}_{;k} - T_{mk}{}^i{}_{;l} + T_{ml}{}^n \cdot T_{nk}{}^i - \\ &\quad -T_{mk}{}^n \cdot T_{nl}{}^i + T_{kl}{}^n \cdot T_{mn}{}^i) \cdot u^m \quad , \end{aligned}$$

$$R_{mk} \cdot u^m \cdot u^k = g^l_i \cdot R^i{}_{mkl} \cdot u^m \cdot u^k = T_{lm}{}^l{}_{;k} \cdot u^k \cdot u^m \quad .$$

Before going on to the kinematic characteristics of a vector field u fulfilling the relations (26) - (28) or (30) - (31), let us now consider some properties of a Weyl's space related to the properties of scalar and vector fields in this type of spaces.

3 Properties of a Weyl's space

1. Parallel transports over Weyl's spaces are at the same time conformal transports. This means that if $\nabla_u \xi = 0$ and $\nabla_u \eta = 0$, then $ul_\xi = (1/2n) \cdot Q_u \cdot l_\xi$, $ul_\eta = (1/2n) \cdot Q_u \cdot l_\eta$, and $u[\cos(\xi, \eta)] = 0$, where $l_\xi :=\mid g(\xi,\xi) \mid^{1/2}$, $l_\eta := \mid g(\eta,\eta) \mid^{1/2}$, $\cos(\xi,\eta) := [g(\xi,\eta)]/(l_\xi \cdot l_\eta)$. If $u = d/d\tau$, then [34]

$$\frac{dl_\xi}{d\tau} = \frac{1}{2n} \cdot Q_u \cdot l_\xi \ , \qquad \frac{dl_\eta}{d\tau} = \frac{1}{2n} \cdot Q_u \cdot l_\eta \quad , \tag{32}$$

and therefore,

$$
\begin{aligned}
l_\xi &= l_{\xi 0}\cdot \exp[\frac{1}{2n}\cdot \int Q_j\cdot dx^j]\ ,\ \ l_\eta = l_{\eta 0}\cdot \exp[\frac{1}{2n}\cdot \int Q_j\cdot dx^j]\ , \qquad (33)\\
l_{\xi 0} &= \text{const.},\, l_{\eta 0} = \text{const.}, \qquad \cos(\xi,\eta) = \text{const. along } \tau(x^k).
\end{aligned}
$$

Furthermore, if $Q_j = -n\cdot \overline{\varphi}_{,j}$ and respectively $Q_u = -n\cdot d\overline{\varphi}/d\tau$, then the equation for l_ξ obtains in this case the simple form

$$\frac{dl_\xi}{d\tau} = -\frac{1}{2}\cdot \frac{d\overline{\varphi}}{d\tau}\cdot l_\xi\ . \qquad (34)$$

The solution for l_ξ could easily be found as

$$l_\xi = l_{\xi 0}\cdot e^{-\frac{1}{2}\cdot \overline{\varphi}} \quad . \qquad (35)$$

The scalar field $\overline{\varphi}$ [as an invariant function $\overline{\varphi}\in \otimes^0{}_0(M)$] appears as a gauge factor changing the length of the vector ξ. This is the reason for calling the scalar field $\overline{\varphi}$ *dilaton field* in a Weyl's space.

2. The metric in a Weyl's space has properties which can be formulated in the following two propositions:

Proposition 1. [35] A metric $\widetilde{g}$ conformal to a Weyl's metric g is also a Weyl's metric. In other words, if $\widetilde{g} = e^\varphi\cdot g$ with $\nabla_\xi g = \frac{1}{n}\cdot Q_\xi\cdot g$ for $\forall \xi\in T(M)$, then $\nabla_\xi \widetilde{g} = \frac{1}{n}\cdot \widetilde{Q}_\xi\cdot \widetilde{g}$.

The proof is trivial.

Therefore, all Weyl's metrics belong to the set of all metrics conformal to a Weyl's metric.

Let the square ds^2 of the line element ds for a Weyl's metric g in $\overline{W}_n$- (or $\overline{Y}_n$)-spaces and in W_n- (or Y_n)-spaces be given in the forms respectively

$$ds^2 = g_{\overline{i}\overline{j}}\cdot dx^i\cdot dx^j \quad , \qquad ds^2 = g_{ij}\cdot dx^i\cdot dx^j \quad .$$

Then, the square $d\widetilde{s}^2$ of the line element $d\widetilde{s}$ for a conformal to the Weyl's metric g will have the forms respectivelly

$$d\widetilde{s}^2 = \widetilde{g}_{\overline{i}\overline{j}}\cdot dx^i\cdot dx^j = e^\varphi.ds^2 \quad , \qquad d\widetilde{s}^2 = \widetilde{g}_{ij}\cdot dx^i\cdot dx^j = e^\varphi.ds^2 \quad .$$

The invariant function $\varphi = \varphi(x^k)\in \otimes^0_0(M)\subset C^r(M)$, $r\geqq 2$, is also called *dilaton field* because of its appearing as a gauge factor changing the line element in a Weyl's space.

For $\pm \widetilde{l}^2_d = \widetilde{g}(d,d) = d\widetilde{s}^2 = \widetilde{g}_{\overline{i}\overline{j}}\cdot dx^i\cdot dx^j$ with $d^i := dx^i$ we have

$$\pm \widetilde{l}^2_d = d\widetilde{s}^2 = e^\varphi.ds^2 = \pm e^\varphi.l^2_d\ .$$

On the other side, by the use of the expression (33) for the length $\widetilde{l}_d$ we find that

$$
\begin{aligned}
\widetilde{l}^2_d &= \widetilde{l}^2_{d0}.\exp[\frac{1}{n}\cdot \int \widetilde{Q}_j\cdot dx^j] = \widetilde{l}^2_{d0}\cdot \exp[\frac{1}{n}\cdot \int (n\cdot \varphi_{,j} + Q_j)\cdot dx^j = \\
&= \widetilde{l}^2_{d0}\cdot \exp\varphi\cdot \exp[\frac{1}{n}\cdot \int Q_j\cdot dx^j] = e^\varphi\cdot l^2_d \quad , \qquad \widetilde{l}^2_{d0} := l^2_{d0} = \text{ const.}
\end{aligned}
$$

For $Q_j = -n \cdot \overline{\varphi}_{,j}$, it follows that

$$\widetilde{l}_d^2 = \widetilde{l}_{d0} \cdot e^{\varphi - \overline{\varphi}} \quad .$$

If $\widetilde{l}_d^2$ does not change along the vector field u, then $\varphi = \overline{\varphi}$ and we have only one dilation field $\varphi = \varphi(x^k) = \overline{\varphi}(x^k)$ which determines the conformal factor of the metric $\widetilde{g}$, conformal to a Weyl's metric g, as well as the Weyl's covector Q. In this case the metric $\widetilde{g}$ appears as a Riemannian metric because of $\widetilde{Q}_j = 0$. If $\varphi \neq \overline{\varphi}$, then the metric $\widetilde{g}$, conformal to g is again a Weyl's metric with $\widetilde{Q}_j = -n \cdot \widetilde{\varphi}_{,j}$ and with $\widetilde{\varphi} = -(\overline{\varphi} - \varphi)$.

Proposition 2. The necessary and sufficient condition for a metric $\widetilde{g}$ conformal ($\widetilde{g} = e^{\varphi} \cdot g$) to a Weyl's metric g [obeying the condition $\nabla_\xi g = \frac{1}{n} Q_\xi \cdot g$ for $\forall \xi \in T(M)$] to be a Riemannian metric $\widetilde{g}$ [obeying the condition $\nabla_\xi \widetilde{g} = 0$ for $\forall \xi \in T(M)$] is the condition

$$Q_\xi = -n \cdot (\xi \varphi) \qquad , \qquad \xi \in T(M) \; , \qquad \varphi \in C^r(M) \; , \; r \geqq 2 \; . \tag{36}$$

The proof is trivial.

Corollary. All Riemannian metrics are conformal to a Weyl's metric in a Weyl's space with $Q_\xi = -n \cdot \xi\varphi$ for $\forall \xi \in T(M)$, $\varphi \in C^r(M)$, $r \geqq 2$, [or in a co-ordinate basis with $Q_k = -n \cdot \varphi_{,k}$].

On the basis of the last proposition we can state that for *every* given Riemannian metric $\widetilde{g}$ from a Riemannian space and a *given scalar (dilaton) field* $\varphi(x^k)$ in this space we could generate a Weyl's metric g in a Weyl's space with the same affine connection as the affine connection in the Riemannian space. Vice versa, for every given Weyl's space with Weyl's covariant vector field Q constructed by a scalar (dilaton) field φ and a Weyl's metric g we could generate a Riemannian metric $\widetilde{g}$ in a Riemannian space with the same affine connection as in the corresponding Weyl's space.

Therefore, *every (metric) tensor-scalar theory of gravitation in a (pseudo) Riemannian space (with or without torsion) could be reformulated in a corresponding Weyl's space with Weyl's metric and dilaton field, generating the Weyl's covector in the Weyl's space* and vice versa: *every (metric) tensor-scalar theory in a Weyl's space with scalar (dilaton) field, generating the Weyl's covector, could be reformulated in terms of a (metric) tensor-scalar theory in the corresponding Riemannian space with the same affine connections as the affine connection in the Weyl's space.*

The kinematic characteristics of a vector field obeying the condition $\nabla_u g - \pounds_u g = 0$ can now be considered from a more general point of view, namely, for spaces with affine connections (which components differ not only by sign) and then specialized for Weyl's spaces.

3.1 Deformation velocity, shear velocity, rotation velocity and expansion velocity

The relative velocity

$$\begin{aligned} {}_{rel}v &= \overline{g}[h_u(\nabla_u\xi)] = g^{ij} \cdot h_{\overline{j}\overline{k}} \cdot \xi^k{}_{;l} \cdot u^l \cdot e_i \quad , \\ e_i &= \partial_i \text{ (in a co-ordinate basis),} \end{aligned} \tag{37}$$

where $\overline{g}[h_u(\xi)] := \xi_\perp = g^{ik} \cdot h_{\overline{k}\overline{l}} \cdot \xi^l \cdot e_i$ is called deviation vector field and (the indices in a co-ordinate and in a non-co-ordinate basis are written in both cases as Latin indices instead of Latin and Greek indices)

$$\begin{gathered} h_u = g - \tfrac{1}{e}.g(u) \otimes g(u) \ , \qquad h_u = h_{ij} \cdot e^i.e^j \ , \qquad \overline{g} = g^{ij} \cdot e_i.e_j , \\ \nabla_u\xi = \xi^i{}_{;j} \cdot u^j \cdot e_i \ , \qquad \xi^i{}_{;j} = e_j\xi^i + \Gamma^i_{kj} \cdot \xi^k \ , \qquad \Gamma^i_{kj} \neq \Gamma^i_{jk} \ , \\ e = g(u,u) = g_{\overline{i}\overline{j}} \cdot u^i \cdot u^j = u_{\overline{i}} \cdot u^i \neq 0 \ , \qquad g(u) = g_{\overline{i}\overline{k}} \cdot u^k \cdot e^i = u_{\overline{i}} \cdot e^i \ , \\ h_{ij} = g_{ij} - \tfrac{1}{e} \cdot u_i \cdot u_j \ , \\ h_u(\nabla_u\xi) = h_{i\overline{j}} \cdot \xi^j{}_{;k} \cdot u^k \cdot e^i \ . \end{gathered} \tag{38}$$

could be written in a $(\overline{L}_n, g)$-space under the conditions $g(u,\xi) := l = 0$, $\pounds_\xi u = 0$, in the form [36], [37]

$$ {}_{rel}v = \overline{g}[d(\xi)] \ . $$

The covariant tensor field d is a generalization for $(\overline{L}_n, g)$-spaces of the well known *deformation velocity* tensor for V_n-spaces [29], [38]. It is usually represented by means of its three parts: the trace-free symmetric part, called *shear velocity* tensor (shear), the anti-symmetric part, called *rotation velocity* tensor (rotation) and the trace part, in which the trace is called *expansion velocity* (expansion) invariant.

After some more complicated as for V_n-spaces calculations, the deformation velocity tensor d can be given in the form [36]

$$d = h_u(k)h_u = h_u(k_s)h_u + h_u(k_a)h_u = \sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u \ , \tag{39}$$

where

$$\begin{gathered} k[g(\xi)] = \nabla_\xi u - T(\xi, u) \ , \quad k = (u^i{}_{;l} - T_{lk}{}^i.u^k).g^{lj}.e_i \otimes e_j = k^{ij}.e_i \otimes e_j \ , \\ k[g(u)] = k(g)u = k^{ij}.g_{\overline{j}\overline{k}}.u^k.e_i = a = \nabla_u u = u^i{}_{;j}.u^j.e_i \ , \end{gathered}$$

$$\begin{aligned} k_s &= {}_sk^{ij} \cdot e_i.e_j ,\ {}_sk^{ij} = \frac{1}{2}(k^{ij} + k^{ji}), \\ {}_ak &= {}_ak^{ij} \cdot e_i \wedge e_j ,{}_a k^{ij} = \frac{1}{2}(k^{ij} - k^{ji}), e_i \wedge e_j = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i) \ . \end{aligned}$$

The tensor σ is the *shear velocity* tensor (shear) ,

$$\begin{gathered}\sigma = {}_sE - {}_sP = E - P - \frac{1}{n-1} \cdot \overline{g}[E-P] \cdot h_u = \sigma_{ij} \cdot e^i . e^j = \\ = E - P - \frac{1}{n-1} \cdot (\theta_o - \theta_1) \cdot h_u \ ,\end{gathered} \tag{40}$$

where

$$\begin{gathered}{}_sE = E - \frac{1}{n-1} \cdot \overline{g}[E] \cdot h_u \ , \qquad \overline{g}[E] = g^{\overline{ij}} \cdot E_{ij} = \theta_o \ , \\ E = h_u(\varepsilon)h_u \ , \qquad k_s = \varepsilon - m \ , \qquad \varepsilon = \frac{1}{2}(u^i{}_{;l} \cdot g^{lj} + u^j{}_{;l} \cdot g^{li}) \cdot e_i . e_j \ , \\ m = \frac{1}{2}(T_{lk}{}^i \cdot u^k \cdot g^{lj} + T_{lk}{}^j \cdot u^k \cdot g^{li}) \cdot e_i . e_j \ .\end{gathered} \tag{41}$$

The tensor ${}_sE$ is the *torsion-free shear velocity* tensor, the tensor ${}_sP$ is the *shear velocity* tensor *induced by the torsion*,

$$\begin{gathered}{}_sP = P - \frac{1}{n-1} \cdot \overline{g}[P] \cdot h_u \ , \qquad \overline{g}[P] = g^{\overline{kl}} \cdot P_{kl} = \theta_1, \qquad P = h_u(m)h_u \ , \\ \theta_1 = T_{kl}{}^k \cdot u^l \ , \quad \theta_o = u^n{}_{;n} - \frac{1}{2e}(e_{,k} \cdot u^k - g_{kl;m} \cdot u^m \cdot u^{\overline{k}} \cdot u^{\overline{l}}) \ , \ \theta = \theta_o - \theta_1 \ .\end{gathered} \tag{42}$$

The invariant θ is the *expansion velocity,* the invariant θ_o is the *torsion-free expansion velocity,* the invariant θ_1 is the *expansion velocity induced by the torsion.*

The tensor ω is the *rotation velocity* tensor (rotation velocity),

$$\begin{gathered}\omega = h_u(k_a)h_u = h_u(s)h_u - h_u(q)h_u = S - Q \ , \\ s = \frac{1}{2}(u^k{}_{;m} \cdot g^{ml} - u^l{}_{;m} \cdot g^{mk}) \cdot e_k \wedge e_l \ , \\ q = \frac{1}{2}(T_{mn}{}^k \cdot g^{ml} - T_{mn}{}^l \cdot g^{mk}) \cdot u^n \cdot e_k \wedge e_l \ , \\ S = h_u(s)h_u \ , \ \ Q = h_u(q)h_u \ .\end{gathered} \tag{43}$$

The tensor S is the *torsion-free rotation velocity* tensor, the tensor Q is the *rotation velocity* tensor *induced by the torsion.*

By means of the expressions for σ, ω and θ the deformation velocity tensor d can be decomposed in two parts: d_0 and d_1

$$d = d_o - d_1 \ , \qquad d_o = {}_sE + S + \frac{1}{n-1} . \theta_o . h_u \ , \qquad d_1 = {}_sP + Q + \frac{1}{n-1} . \theta_1 . h_u \ , \tag{44}$$

where d_o is the *torsion-free deformation velocity* tensor and d_1 is the *deformation velocity* tensor *induced by the torsion.* For the case of V_n-spaces $d_1 = 0$ (${}_sP = 0$, $Q = 0$, $\theta_1 = 0$).

After some calculations, the shear velocity tensor σ and the expansion velocity θ can also be written in the forms

$$\sigma = \frac{1}{2}\{h_u(\nabla_u \overline{g} - \pounds_u \overline{g})h_u - \frac{1}{n-1} \cdot (h_u[\nabla_u \overline{g} - \pounds_u \overline{g}]) \cdot h_u\} =$$

$$= \frac{1}{2}\{h_{i\overline{k}} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{\overline{l}j} - \frac{1}{n-1} \cdot h_{\overline{kl}} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{ij}\} \cdot e^i.e^j \ , \tag{45}$$

$$\theta = \frac{1}{2} \cdot h_u[\nabla_u \overline{g} - \pounds_u \overline{g}] = \frac{1}{2} h_{\overline{ij}} \cdot (g^{ij}{}_{;k} \cdot u^k - \pounds_u g^{ij}) \ . \tag{46}$$

The physical interpretation of the velocity tensors d, σ, ω, and of the invariant θ for the case of V_4-spaces [39], [28] , can also be extended for $(\overline{L}_4, g)$-spaces. It is easy to be seen that the existence of some kinematic characteristics (${}_sP$, Q, θ_1) depends on the existence of the torsion tensor field. They vanish if it is equal to zero (e.g. in V_n-spaces). It should be stressed that the decomposition of the deformation tensor d could not follow from the decomposition of $u^i{}_{;j}$ as this has been done by Ehlers [28] for (pseudo) Riemannian spaces without torsion (V_n-spaces, $n = 4$). In $(\overline{L}_n, g)$-spaces

$$\begin{aligned} u^i{}_{;j} \quad &= \quad \frac{1}{e} \cdot a^i \cdot u_{\overline{j}} + g^{ik} \cdot ({}_sE_{\overline{kj}} + S_{\overline{kj}} + \frac{1}{n-1} \cdot \theta_0 \cdot h_{\overline{kj}}) + \\ &\quad + \frac{1}{2 \cdot e} \cdot u^i \cdot (e_{,l} - g_{mn;l} \cdot u^{\overline{m}} \cdot u^{\overline{n}}) \cdot h^{lk} \cdot g_{\overline{kj}} \quad . \end{aligned} \tag{47}$$

The representation of the shear velocity tensor σ and the expansion invariant θ by means of the covariant and Lie derivatives of the contravariant metric tensor $\overline{g}$ give rise to some important conclusions about their vanishing or nonvanishing in a $(\overline{L}_n, g)$-space.

From the structure of σ and θ in the last two expression, it is obviously that if $\pounds_u \overline{g} = \nabla_u \overline{g}$ then $\sigma = 0$ and $\theta = 0$, i.e. the condition $\pounds_u \overline{g} = \nabla_u \overline{g}$ appears as a sufficient conditions for $\sigma = 0$ and $\theta = 0$. On the other side, this condition could be written in the form $\pounds_u g = \nabla_u g$ because of the relations $\pounds_u \overline{g} = -\overline{g}(\pounds_u g)\overline{g}$ and $\nabla_u \overline{g} = -\overline{g}(\nabla_u g)\overline{g}$. For Weyl's spaces ($\nabla_u g = \frac{1}{n} \cdot Q_u \cdot g$) the same condition degenerate in the condition for the existence of a conformal Killing vector u

$$\pounds_u g = \lambda_u \cdot g \qquad \text{with} \qquad \lambda_u = \frac{1}{n} \cdot Q_u \ . \tag{48}$$

On the basis of the above considerations we could now formulate the following propositions:

Proposition 3. If a metric g in a space with affine connections and metrics [a $(\overline{L}_n, g)$- or a (L_n, g)-space] fulfills the condition

$$\pounds_u \overline{g} = \nabla_u \overline{g} \quad \text{or} \qquad \pounds_u g = \nabla_u g \ , \tag{49}$$

then the space admits a non-null shear-free and expansion-free contravariant vector field u.

Proposition 4. If a contravariant non-null vector field fulfills in a Weyl's space a conformal Killing equation of the type

$$\pounds_u g = \lambda_u \cdot g \qquad \text{with} \qquad \lambda_u = \frac{1}{n} \cdot Q_u \ , \tag{50}$$

then this conformal vector field u is also a shear-free and expansion-free vector field.

Proposition 5. If a contravariant non-null vector field u in a space with affine connections and metrics [a $(\overline{L}_n, g)$- or a (L_n, g)-space] fulfills the equation (30)

$$u^k{}_{;j} - T_{jl}{}^k \cdot u^l = 0 \qquad \text{or} \qquad u^k{}_{;j} = T_{jl}{}^k \cdot u^l \ ,$$

then this vector field is an auto-parallel shear-free and expansion-free vector field.

Proposition 6. If a contravariant non-null vector field u in a Weyl's space fulfills the equation (30)

$$u^k{}_{;j} - T_{jl}{}^k \cdot u^l = 0 \quad ,$$

then it is an auto-parallel, shear-free and expansion-free conformal Killing vector field.

The auto-parallel equation (6) for the vector field u is interpreted as an equation of motion for a free spinless test particle in spaces with affine connection and metrics [22], [23]. Let us now consider this equation more closely in Weyl's spaces.

4 Auto-parallel equation in Weyl's spaces as an equation for a free moving spinless test particles

Usually the following definition of a free moving test particle in a space with affine connections and metrics [and especially in (pseudo) Riemannian spaces without torsion] is introduced [40]:

Definition 3. A free spinless test particle is a material point with rest mass (density) ρ, velocity u (as tangent vector u to its trajectory) and momentum (density) $p := \rho \cdot u$ with the following properties:

(a) The momentum density p does not change its direction along the world line of the material point, i.e. the vector p fulfills the recurrent condition $\nabla_u p = f \cdot p$, or the condition $\nabla_u p = 0$, as conditions for parallel transport along u.

(b) The momentum density p does not change its length $l_p = \mid g(p,p) \mid^{1/2}$ along the world line of the material point, i.e. p fulfills the condition $ul_p = 0$.

The change of the length of a vector field p along a vector field u in a $(\overline{L}_n, g)$-space could be found in the form [34]

$$ul_p = \pm \frac{1}{2 \cdot l_p} \cdot [(\nabla_u g)(p,p) + 2 \cdot g(\nabla_u p, p)] \quad , \qquad l_p \neq 0 \ . \tag{51}$$

Let us now consider the two conditions (a) and (b) for p separately to each other.

(a) If we write p in its explicit form $p = \rho \cdot u$, then the condition for a parallel transport of p along u could be written as

$$\nabla_u u = [f - u(\log \rho)] \cdot u \ , \tag{52}$$

with

$$f = u(\log \rho) + \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u, u)] \ , \qquad e = g(u, u) \neq 0 \ , \tag{53}$$

and

$$\nabla_u u = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \ . \tag{(a)}$$

In the special case of (pseudo) Riemannian spaces (V_n- or U_n- spaces), where $\nabla_u g = 0$, $e =$ const. $\neq 0$, $\rho =$const., $f = 0$, it follows that $\nabla_u p = 0$, and $\nabla_u u = 0$. At the same time, $ul_p = 0$. The parallel equation $\nabla_u p = 0$ has as a corollary the preservation of the length l_p of p along u. This is not the case if a space is not a (pseudo) Riemannian space.

(b) The conservation of the momentum density p along the trajectory of the particles $[ul_p = 0]$ requires the transport of p on this trajectory to be of the type of a Fermi-Walker transport, i.e. p should obey an equation of the type [17], [19]

$$\nabla_u p = \overline{g}({}^F\omega - \frac{1}{2} \cdot \nabla_u g)(p) = \overline{g}[{}^F\omega(p)] - \frac{1}{2} \cdot \overline{g}(\nabla_u g)(p) \ , \tag{54}$$

where ${}^F\omega \in \Lambda^2(M)$ is an antisymmetric tensor of 2nd rank. For a free particle it could be related to the rotation velocity tensor (43) of the velocity u, i.e. ${}^F\omega := \omega$. Then $\omega(p) = 0$ [because of $\omega(\rho \cdot u) = \rho \cdot (\omega(u)) = 0$] and we have for $\nabla_u p$

$$\nabla_u p = -\frac{1}{2} \cdot \overline{g}(\nabla_u g)(p) \ , \qquad ul_p = 0 \ . \tag{55}$$

For the vector field u follows the corresponding condition

$$\nabla_u u = -\{[u(\log \rho)] \cdot u + \frac{1}{2} \cdot \overline{g}(\nabla_u g)(u)\} \ . \tag{(b)}$$

Therefore, for u we have two equations as corollaries from the requirements for the momentum density p: equation (a) which follows from the condition for preservation of the direction of the momentum density p, and equation (b) which follows from the condition for preservation of the length l_p of the momentum density p.

(a) The first equation (a) for u is the auto-parallel equation in its non-canonical form. It does not depend on the rest mass density ρ of the particle. From the equation, it follows that the necessary and sufficient condition for a spinless test particle to move in space with affine connections and metrics on a trajectory described by the auto-parallel equation in its canonical form ($\nabla_u u = 0$) is the condition

$$[ue - (\nabla_u g)(u,u)] \cdot u = 0 \quad , \qquad e \neq 0 \ . \tag{56}$$

Since $g(u,u) = e \neq 0$, after contracting the equation with $g(u)$, we obtain the condition

$$ue - (\nabla_u g)(u,u) = 0 \ , \qquad \text{or} \qquad ue = (\nabla_u g)(u,u) \quad . \tag{57}$$

This condition determines how the length of the vector u should change with respect to the change of the metric g along u if u should be an auto-parallel vector field with $\nabla_u u = 0$.

If we consider a Weyl's space as a model of space-time, this condition will take the form

$$ue = \frac{1}{n} \cdot Q_u \cdot e \quad , \text{or} \qquad u(\log e) = \frac{1}{n} \cdot Q_u \quad , \tag{58}$$

leading to the relation for e

$$e = e_0 \cdot \exp(\frac{1}{n} \cdot \int Q_u \cdot d\tau) \quad , \qquad\qquad e_0 = \text{ const.}, \tag{59}$$

where $u = d/d\tau$ and $\tau = \tau(x^k)$ is the canonical parameter of the trajectory of the particle.

If Q_u is constructed by the use of a dilaton field $\overline{\varphi}$ as $Q_u = -d\overline{\varphi}/d\tau$, then e would change under the condition

$$e = e_0 \cdot \exp(-\frac{1}{n} \cdot \overline{\varphi}) \ . \tag{60}$$

The dilaton field $\overline{\varphi}$ could be represented by means of e in the form

$$\overline{\varphi} = -n \cdot \log(\frac{e}{e_0}) \ . \tag{61}$$

Therefore, the dilaton field $\overline{\varphi}$ takes the role of a length scaling factor for the velocity of a test particle.

(b) From the second equation (b) for u, it follows that a necessary and sufficient condition for a free spinless test particle to move in a space with affine connections and metrics on a trajectory, described by the auto-parallel equation in its canonical form ($\nabla_u u = 0$), is the condition

$$[u(\log \rho)] \cdot u = -\frac{1}{2} \cdot \overline{g}(\nabla_u g)(u) \quad . \tag{62}$$

Since $g(u,u) = e \neq 0$, after contracting the last equation with $g(u)$ we obtain the condition

$$u(\log\rho)\cdot e = -\frac{1}{2}\cdot\overline{g}(\nabla_u g)(u)[g(u)] = -\frac{1}{2}\cdot(\nabla_u g)(u,u) \quad ,$$

or

$$u(\log\rho) = -\frac{1}{2\cdot e}\cdot(\nabla_u g)(u,u) \ . \tag{63}$$

For $u = d/d\tau$, it follows the equation

$$\frac{d}{d\tau}(\log\rho) = -\frac{1}{2\cdot e}\cdot(\nabla_u g)(u,u) \ ,$$

with the solution for $\rho(x^k(\tau))$

$$\rho = \rho_0\cdot\exp(-\frac{1}{2}\cdot\int\frac{1}{e}\cdot(\nabla_u g)(u,u)\cdot d\tau) \quad .$$

The last condition is for the rest mass density ρ which it has to obey if the particle should move on an auto-parallel trajectory or if we observe the motion of a particle as a free motion in the corresponding space considered as a mathematical model of the space-time.

If we consider a Weyl's space as a model of the space-time, the condition (62) will take the form

$$[u(\log\rho)]\cdot u = -\frac{1}{2\cdot n}\cdot Q_u\cdot u \ , \tag{64}$$

or the form

$$[u(\log\rho) + \frac{1}{2\cdot n}\cdot Q_u]\cdot u = 0 \ .$$

Since $g(u,u) = e \neq 0$, after contracting the last equation with $g(u)$ we obtain the condition

$$u(\log\rho) + \frac{1}{2\cdot n}\cdot Q_u = 0 \quad . \tag{65}$$

Therefore, the rest mass density ρ should change on the auto-paralel trajectory of the particle as

$$\rho = \rho_0\cdot\exp\left[-\frac{1}{2\cdot n}\cdot\int Q_u\cdot d\tau\right] \ , \qquad \rho_0 = \text{const.} \tag{66}$$

Furthermore, if Q_u is constructed by the use of a dilaton field $\overline{\varphi}$ as $Q_u = -d\overline{\varphi}/d\tau$, then ρ would change under the condition

$$\rho = \rho_0\cdot\exp\left[\frac{1}{2\cdot n}\cdot\int\frac{d\overline{\varphi}}{d\tau}\cdot d\tau\right] = \rho = \rho_0\cdot\exp(\frac{1}{2\cdot n}\cdot\overline{\varphi}) \ . \tag{67}$$

The dilaton field $\overline{\varphi}$ could be represented by means of ρ in the form

$$\overline{\varphi} = 2 \cdot n \cdot (\log \frac{\rho}{\rho_0}) \ . \tag{68}$$

Therefore, the dilaton field $\overline{\varphi}$ takes the role of a mass density scaling factor for the rest mass density of a test particle. This is another physical interpretation as the interpretation used by other authors as mass field, pure geometric gauge field and etc.

Since $\overline{\varphi} = -n \cdot \log(e/e_0) = 2n \cdot \log(\rho/\rho_0)$, a relation between e and ρ follows in the form

$$\rho^2 \cdot e = \rho_0^2 \cdot e_0 = \text{ const. } = l_p^2 \ , \tag{69}$$

which is exactly the condition (b) of the definition for a free moving spinless test particle.

5 Conclusion

In the present paper the conditions are found under which a space with affine connections and metrics and especially a Weyl's space admit shear-free and expansion-free non-null vector fields. In a Weyl's space the vector fields appear as conformal Killing vector fields. In such type of spaces only the rotation velocity is not vanishing. This fact could be used as a theoretical basis for models in contunuous media mechanics and in the modern gravitational theories, where a rotation velocity could play an important role. Further, necessary and sufficient conditions are found under which a free spinless test particle could move in spaces with affine connections and metrics on an auto-parallel curve. In Weyl's spaces with Weyl's covector, constructed by the use of a dilaton field, the dilaton field appears as a scaling factor for the rest mass density as well as for the velocity of the test particle. The last fact leads to a new physical interpretation of a dilaton field in classical field theories over spaces with affine connections and metrics and especially over Weyl's spaces.

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